PHYC 511
Spring 2019

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Problem Session 3 01/30/2019

(1) Problem 3.9, Jackson.

(2) Problem 3.12 (a), (b), Jackson.

(1) Vanishing the potential at $z=0$ and $z=L$ requires that:

$$Z(z) \propto \sin\left(\frac{n\pi}{L}z\right) \Rightarrow k_n^2 = \frac{-n^2\pi^2}{L^2}$$

For $k^2<0$, the potential involves modified Bessel functions I_m, K_m.

However, since K_m blows up at $\rho=0$, we have the following general form for the potential:

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, I_m(k_n\rho)\, e^{im\phi} \sin\left(\frac{n\pi}{L}z\right)$$

The boundary condition at $\rho=a$ implies that:

$$\sum_{m,n} A_{mn}\, I_m(k_n a)\, e^{im\phi} \sin\left(\frac{n\pi}{L}z\right) = V(\phi,z)$$

This is a double Fourier series over ϕ and z. Then:

$$\boxed{A_{mn} = \frac{1}{\pi L\, I_m\left(\frac{n\pi}{L}a\right)} \int_0^L dz \int_0^{2\pi} d\phi\, V(\phi,z)\, e^{-im\phi} \sin\left(\frac{n\pi}{L}z\right)}$$

Substituting this in the above expression for Φ gives the requested series expansion.

(2) (a) First of all, due to azimuthal symmetry, there is no ϕ-dependence in the potential. Also, since the potential is due to a localized distribution, it must vanish at infinity. For $z>0$, this requires that $Z(z) \propto e^{-kz}$ where $k>0$.

As a result, the most general form of the potential is:

$$\boxed{\Phi(\rho,z) = \int_0^\infty A(k)\, J_0(k\rho)\, e^{-kz}\, k\, dk}$$

(b) For $z=0$, we have:

$$\Phi(\rho,0) = \int_0^\infty A(k)\, J_0(k\rho)\, k\, dk = \begin{cases} V & \rho<a \\ 0 & \rho>a \end{cases}$$

Using the Hankel transform relation gives:

$$A(k) = \int_0^\infty \Phi(\rho,0)\, J_0(k\rho)\, \rho\, d\rho = \frac{V}{k}\int_0^a \underbrace{J_0(k\rho)\, k\rho\, d\rho}_{\frac{d}{d(k\rho)}\left(J_1(k\rho)\, k\rho\right)} = \frac{Va}{k} J_1(ka)$$

Therefore:

$$\boxed{\Phi(\rho,z) = Va\int_0^\infty J_1(ka)\, J_0(k\rho)\, e^{-kz}\, dk}$$

For $\rho = 0$, we have $J_0(k\rho) = 1$. Hence:

$$\Phi(0,z) = Va \underbrace{\int_0^{\infty} J_1(ka)\, e^{-kz}\, dk}_{\frac{1}{a}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)}$$

← from integral tables

This results in:

$$\boxed{\Phi(0,z) = V\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)}$$